PENETRATION OF A WEDGE INTO COMPRESSIBLE HALF-SPACE

(PRONIKANIE KLINA V SZHIMAEMOE POLUPROSTRANSTVO)

PMM Vol.25, No.3, 1961, pp. 485-489

Z.N. DOBROVOL'SKAIA
(Moscow)

(Received October 31, 1960)

The problem on the penetration of a wedge into an incompressible fluid was considered in [1] through [5]. The analogous problem taking compressibility into account was solved for the case of a thin wedge [6].

In the present paper the plane problem is considered, regarding the immersion into a compressible liquid of a wedge of finite apex angle 2α with a constant velocity v_0, which is assumed to be small as compared to the sound velocity a in the medium. The liquid is assumed to be ideal and weightless. In its exact formulation this problem is nonlinear and so far has not been solved.

Under the condition of smallness of the penetration velocity, however, the changes in all hydrodynamic quantities in the given problem will be small, on the strength of which they may be treated in acoustic approximation.

The problem considered is of interest, because under realistic conditions of penetration we deal with bodies of finite dimensions, when their idealization as thin blades becomes invalid.

In treating the problem on the penetration of a thin wedge, the condition of flow of the solid surface is referred to the axis of the wedge. In the case of a finite wedge angle, however, this condition is not satisfied, which leads to a considerable complication of the boundary condition in the region of perturbed motion in the plane of Chaplygin variables. The problem is solved by means of conformal mapping of the region of perturbed motion on the upper half-plane and by reduction to a Hilbert boundary-value problem with discontinuous coefficients.

1. Let the apex of the wedge touch the free surface at the instant of time $t = 0$. For any $t > 0$ we will have in the region of perturbed motion, bounded by a Mach wave, the free surface and the wedge surface:

$$u = u_1(x, y, t) + \ldots, \qquad v = v_1(x, y, t) + \ldots$$
$$p = p_0 + p_1(x, y, t) + \ldots, \qquad \rho = \rho_0 + \rho_1(x, y, t) + \ldots \tag{1.1}$$

Here x, y are fixed axes of Cartesian coordinates with origin at the point of contact of wedge apex and the free surface, t is the time, p_0 and ρ_0 are the pressure and density, respectively, of the liquid at rest.

Let us introduce self-similar variables $x_1 = x/at$, $y_1 = y/at$. Then

$$u_1(x, y, t) = v_0\bar{u}_1(x_1, y_1), \qquad p_1(x, y, t) = \rho_0 v_0^2 \bar{p}_1(x_1, y_1)$$
$$v_1(x, y, t) = v_0\bar{v}_1(x_1, y_1), \qquad \rho_1(x, y, t) = \rho_0\bar{\rho}_1(x_1, y_1)$$

Let us substitute Expressions (1.1) into equations which describe the plane nonsteady motion of an ideal compressible liquid, and let us eliminate $\bar{u}_1$, $\bar{v}_1$ and $\bar{\rho}_1$; as a result of linearization we obtain

$$(1 - x^2)\frac{\partial^2 p}{\partial x^2} - 2xy\frac{\partial^2 p}{\partial x \partial y} + (1 - y^2)\frac{\partial^2 p}{\partial y^2} - 2y\frac{\partial p}{\partial y} - 2x\frac{\partial p}{\partial x} = 0 \tag{1.2}$$

Here and in the sequel the bar and the subscript of dimensionless quantities will be omitted.

The problem consists in determining the function $p(x, y)$ satisfying in the region $ABCD$ Equation (1.2) and the following conditions on the boundary:

$$\frac{\partial p}{\partial n} = 0 \quad \text{on } AB, \quad \frac{\partial p}{\partial y} = 0 \quad \text{on } AD$$
$$p = 0 \quad \text{on } DC, \quad p = 0 \quad \text{on } BC \tag{1.3}$$

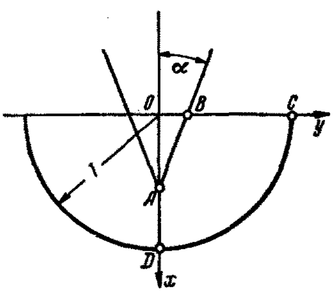

Fig. 1.

As the wedge penetrates, the free surface is deformed primarily in the region adjacent to the edge of the wedge. In the present study this effect is not taken into account, and the condition of constancy of pressure is referred to the corresponding portion of the y-axis. The treatment of this problem is a necessary step, because later on it will permit the deformation of the free surface to be taken into account approximately.

2. Let us pass from the variable $z = re^{i\theta}$, $(z = x + iy)$ into the plane of the variable $\zeta = \epsilon e^{i\theta}$, $(\zeta = \xi + i\eta)$ with the aid of the Chaplygin transformation

$$r = \frac{2\varepsilon}{1+\varepsilon^2}, \qquad \theta = \tan^{-1}\frac{y}{x} \tag{2.1}$$

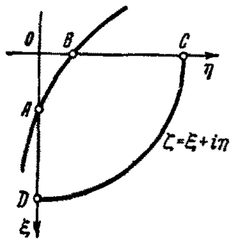

Fig. 2.

Equation (1.2) for the pressure will then be transformed into a Laplace equation in polar coordinates, the region $ABCD$ of the plane z into a curvilinear quadrangle (Fig. 2), bounded by arcs of circles AB and DC and segments of axes ξ and η, and instead of conditions (1.3) we will have

$$\frac{\partial p}{\partial \eta} = 0 \quad \text{on } AD, \qquad p = 0 \quad \text{on } DC \text{ and } CB$$

$$\frac{\partial p}{\partial \xi}\left[(1+\xi^2-\eta^2)\sin\alpha + 2\xi\eta\cos\alpha\right] + \frac{\partial p}{\partial \eta}\left[(1-\xi^2+\eta^2)\cos\alpha + 2\xi\eta\sin\alpha\right] = 0 \quad \text{on } BA \tag{2.2}$$

(For clarity the corresponding points in different planes are indicated by like letters.) Thus the problem was reduced to the determination of the harmonic function $p(\xi, \eta)$ in the region $ABCD$ with boundary conditions (2.2).

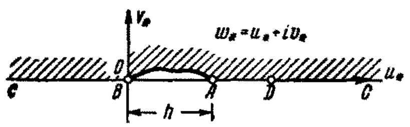

Fig. 3.

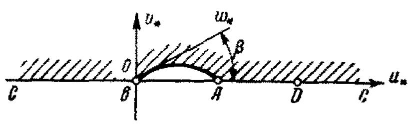

Fig. 4.

3. The function

$$w_* = \frac{1}{1-M^2\tan^2\alpha} - \left(\frac{\zeta^2-1}{\zeta^2+1}\right)^2 \tag{3.1}$$

will transform the region considered into the upper half-plane, with the exclusion of the segment BA (Fig. 3). The equations of the curve BA could not be obtained in explicit form and for this reason the region $ABCD$ of the plane (w_*) is mapped on the upper half-plane approximately. To this end we replace the curve BA (it can be shown that it is sufficiently monotonic) by a circular arc passing through the points A and B in such a manner that the mean quadratic deviation of the ordinates of the sought curve be a minimum (Fig. 4).

The function

$$w = \frac{h\pi}{\pi-\beta}\left[1-\left(1-\frac{h}{w_*}\right)^{\frac{\pi}{\pi-\beta}}\right]^{-1} \qquad (3.2)$$

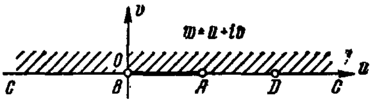

Fig. 5.

where β is the inclination of the tangent to the approximating circle at the origin of coordinates, will map the upper half-plane with the exclusion of the circular segment on the upper half-plane (Fig. 5).

The boundary condition (2.2) will be transformed to the following condition, satisfied on the real u-axis and the w-plane:

$$\begin{gathered}\frac{\partial p}{\partial u}\operatorname{Re}\left[iw'(\zeta)\left(e^{-i\alpha}-\zeta^2 e^{i\alpha}\right)\right]+\frac{\partial p}{\partial v}\operatorname{Im}\left[iw'(\zeta)\left(e^{-i\alpha}-\zeta^2 e^{i\alpha}\right)\right]=0 \text{ on } BA \\ \frac{\partial p}{\partial v}=0 \quad \text{on } AD, \qquad p=0 \quad \text{on } DC, \qquad p=0 \quad \text{on } CB\end{gathered} \qquad (3.3)$$

Here $w'(\zeta)$ is the derivative of the mapping function, and $\zeta(u)$ is a function, inverse to (3.1), (3.2).

4. The harmonic function $p(u, v)$ may be considered as the real part of some function $f(w) = p + iq$, analytic in the upper half-plane. Let us introduce $f_1(w) = p_1 + iq_1 = f'(w)$. Then the boundary condition (3.3) for the function $f_1(w)$ will take on the form

$$a_1(u)\,p_1 + b_1(u)\,q_1 = 0 \qquad (4.1)$$

Here

$$\begin{array}{lll} a_1 = 1, \quad b_1 = 0 & & (-\infty < u \leqslant 0) \\ a_1 = \operatorname{Re}\left[iw'(\zeta)\left(e^{-i\alpha}-\zeta^2 e^{i\alpha}\right)\right] & & \\ b_1 = -\operatorname{Im}\left[iw'(\zeta)\left(e^{-i\alpha}-\zeta^2 e^{i\alpha}\right)\right] & & \text{for } \zeta=\zeta(u),\ 0 \leqslant u \leqslant u_A \\ a_1 = 0, \quad b_1 = 1 & & (u_A \leqslant u \leqslant u_D) \\ a_1 = 1, \quad b_1 = 0 & & (u_D \leqslant u < +\infty) \end{array} \qquad (4.2)$$

The boundary-value problem for the function $f_1(w)$ formulated above is a homogeneous Hilbert problem for the upper half-plane with discontinuous coefficients. It has a non-unique solution; the character of the solution depends upon the type of singularities admitted at the points of discontinuity of the coefficients of the boundary conditions. In the problem considered the real part of the integral

$$J(w) = \int f_1(w)\,dw$$

is the pressure; therefore, the function $f_1(w)$ must satisfy the following conditions:

1. The function $f_1(w)$ must be regular in the upper half-plane, excluding the real axis.
2. At the points B and D of discontinuity of coefficients $a_1(u)$ and $b_1(u)$ the integral $J(w)$ must be bounded.
3. At infinity $J(w)$ behaves as cw^{-1}.
4. At the point A a singularity of the function $f_1(w)$ is admitted, and the character of the singularity is determined by requirements 2 and 3.

The solution of the homogeneous Hilbert problem is given by the function [7]

$$f_1(w) = \exp\left[\frac{1}{\pi}\int_{-\infty}^{+\infty} \tan^{-1} \frac{b_1(u)}{a_1(u)}\frac{du}{u-w}\right] iP(w)\left[\prod_n (w-\nu_n)\right]^{-1} \tag{4.3}$$

where $P(w)$ is a polynomial which takes on real values on the real axis, while ν_n are the abscissas of the points of discontinuity of the function $\tan^{-1}[b_1(u)/a_1(u)]$.

The function $P(w)$ and the number of values ν_n are selected in accordance with the class in which the solution is sought.

In the multivalued function $\tan^{-1}(b_1/a_1)$ we isolate those branches which are in the first and fourth quarter.

We note [8] that at the point of discontinuity ν the functions are

$$\omega(u) = \tan^{-1}[b_1(u)/a_1(u)]$$

$$\exp\left[\frac{1}{\pi}\int_{-\infty}^{+\infty} \omega(u)\frac{du}{u-w}\right] = (w-\nu)^{\frac{\omega(\nu-0)-\omega(\nu+0)}{\pi}} e^{2i\Phi_0(w)} \tag{4.4}$$

and at infinity

$$\exp\left[\frac{1}{\pi}\int_{-\infty}^{+\infty} \omega(u)\frac{du}{u-w}\right] = 1 \tag{4.5}$$

In the problem considered

$$-\frac{1}{2} < \frac{\omega(B-0)-\omega(B+0)}{\pi} < 0, \qquad -\frac{1}{2} < \frac{\omega(A-0)-\omega(A+0)}{\pi} < 0$$
$$\frac{\omega(D-0)-\omega(D+0)}{\pi} = \frac{1}{2} \tag{4.6}$$

Therefore, to satisfy conditions 1 through 4, we must choose

$$P(w) = C_1, \qquad \prod_n (w-\nu_n) = (w-u_D)(w-u_A)$$

Substituting these functions, as well as the values of the coefficients $a_1(u)$ and $b_1(u)$ from (4.2) into Expression (4.3), we finally obtain

$$f_1(w) = iC_1(u_D-w)^{-1/2}(u_A-w)^{-1/2}\times$$
$$\times \exp\left\{\frac{1}{\pi}\int_0^{u_A} \tan^{-1}\frac{-\operatorname{Im}[iw'(\zeta)(e^{-i\alpha}-\zeta^2e^{i\alpha})]_{\zeta=\zeta(u)}}{\operatorname{Re}[iw'(\zeta)(e^{-i\alpha}-\zeta^2e^{i\alpha})]_{\zeta=\zeta(u)}}\frac{du}{u-w}\right\} \tag{4.7}$$

Here u_A and u_D are the abscissas of the points of discontinuity A and D, while C is a real constant which is determined from the condition that at the face of the wedge the projections of the velocity of the fluid particles and of the velocity of wedge penetration must be equal

$$\frac{C_1M}{\sin\alpha}\int_{1/\sqrt{2}}^{M(1+\cot\alpha)^{-1}}\left[\frac{\partial p(x,y)}{\partial x}\sin\alpha+\frac{\partial p(x,y)}{\partial y}\cos\alpha\right]_{x=y=\mu}\frac{d\mu}{\mu}=1 \qquad \left(M=\frac{v_0}{a}\right)$$

The expression for the pressure is of the form

$$p(u,v) = \operatorname{Re}\int f_1(w)\,dw \tag{4.8}$$

It is easily shown that the integration constant is equal to zero. Expressions (4.7), (4.8) represent the solution of the problem in the w-plane.

The function $p(u, v)$ is regular everywhere in the upper half-plane and on the real axis, with the exception of a point A with abscissa u_A, where there exists a singularity of an integrable type. In the physical plane the apex of the wedge corresponds to this point.

5. Let us consider a thin wedge. The solution (4.7), (4.8) is valid in this case for any subsonic penetration velocity. Let α in the preceding formulas approach zero. Then

$$f_1(w) = C_1(M^2-w)^{-1}w^{-1/2}(1-w)^{-1/2}$$

At the face of the wedge

$$AB\,(v = 0,\quad 0 \leqslant u \leqslant M^2)\;\; u = x^2.$$

Therefore we will have for the pressure distribution along the wedge

$$p - p_0 = C_1 \rho_0 v_0{}^2 M^{-1} (1 - M^2)^{-1/2} \ln \frac{M\sqrt{1-x^2} + x\sqrt{1-M^2}}{M\sqrt{1-x^2} - x\sqrt{1-M^2}} \qquad (0 \leqslant x \leqslant M) \tag{5.1}$$

Formula (5.1) coincides with the solution obtained in [6].

In the case of a thin wedge the formula for the resistive force of the wedge during penetration into an incompressible fluid coincides with the results of [6]. In the case of a wedge with a finite angle, only a numerical comparison with the available results is possible, since the function $w(\zeta)$ which maps the region $ABCD$ on the upper half-plane is found approximately.

We note that this function may be found also in another manner, different form the one suggested in the present paper. The form of the solution, however, namely Formula (4.7), does not change.

The author is grateful to L.A. Galin for valuable advice.

BIBLIOGRAPHY

1. Sedov, L.I., Ob udare tverdogo tela, plavaiushchego na poverkhnosti neszhimaemoi zhidkosti (On the impact of a rigid body, floating on the surface of an incompressible fluid). *Trudy TsAGI* No. 187, 1934.

2. Sedov, L.I., *Metody podobiia i razmernosti v mekhanike (Methods of Similarity and Dimensions in Mechanics)*. Gostekhizdat, 1957.

3. Wagner, H., Über Stoss- und Gleitvorgänge an der Oberfläche von Flüssigkeiten. *ZAMM* Nr. 4, pp. 194-215, 1932.

4. Borisova, E.P., Korianov, P.P. and Moiseev, N.N., Ploskie i osesimmetrichnye avtomodel'nye zadachi pogruzheniia i soudareniia strui (Plane and axially-symmetric self-similar problems of immersion and mutual impact of jets). *PMM* Vol. 23, No. 2, 1959.

5. Borg, S.F., Some contributions to the wedge-water entry problem. *Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers* Vol. 83, NEM2, April 1957.

6. Sagomonian, A.Ia., Pronikanie uzkogo klina v szhimaemuiu zhidkost' (Penetration of a thin wedge into a compressible fluid). *Vestnik MGU* No. 2, 1956.

7. Galin, L.A., *Kontaktnye zadachi teorii uprugosti (Contact Problems of the Theory of Elasticity)*. Gostekhteoretizdat, 1953.

8. Gakhov, F.D., *Kraevye zadachi (Boundary-value Problems)*. Fizmatgiz, 1958.

Translated by G.H.